2nd Semester, Study Material

Engineering Mathematics - II

Th 3. Engineering mathematics-II

Integration

Integration:

The process of finding functions whose derivative is given, is called anti-differentiation or Integration.

Let $F(x)$ be a function.

If $\frac{d}{dx}\{\phi(x)\} = F(x)$, then

$\int F(x) \cdot dx = \phi(x) + c$

Where c is called the constant of integration or arbitrary constant.

Symbols

$F(x) \rightarrow$ Integrand

$F(x) \cdot dx \rightarrow$ Elements of integration

$\int \rightarrow$ sign of integration

$\phi(x) \rightarrow$ Anti-derivative or integral of function $F(x)$

Scanned with CamScanner

Some Standard Formulae of Integration :

i) $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c, \ (n \neq -1)$

ii) $\int 1/x \cdot dx = \log_e |x| + c$

iii) $\int a^x \cdot dx = \frac{a^x}{\log_e a} + c$

iv) $\int e^x \cdot dx = e^x + c$

v) $\int \sin x \cdot dx = -\cos x + c$

vi) $\int \cos x \cdot dx = \sin x + c$

vii) $\int \sec^2 x \cdot dx = \tan x + c$

viii) $\int \operatorname{cosec}^2 x \cdot dx = -\cot x + c$

ix) $\int \sec x \cdot \tan x \cdot dx = \sec x + c$

x) $\int \operatorname{cosec} x \cdot \cot x \cdot dx = -\operatorname{cosec} x + c$

xi) $\int \frac{1}{\sqrt{1-x^2}} \cdot dx = \sin^{-1} x + c$

xii) $\int \frac{-1}{\sqrt{1-x^2}} \cdot dx = \cos^{-1} x + c$

xiii) $\int \frac{1}{1+x^2} \cdot dx = \tan^{-1} x + c$

xiv) $\int \frac{-1}{1+x^2} \cdot dx = \cot^{-1} x + c$

xv) $\int \frac{1}{x\sqrt{x^2-1}} \cdot dx = \sec^{-1} x + c$

xvi) $\int \frac{-1}{x\sqrt{x^2-1}} \cdot dx = \operatorname{cosec}^{-1} x + c$

Q.1 : Evaluate

i) $\int \left(5x^3 + 2x^5 - 7x + \frac{1}{\sqrt{x}} + \frac{5}{x}\right) \cdot dx$

<u>soln</u>:

we have $\int \left(5x^3 + 2x^5 - 7x + \frac{1}{\sqrt{x}} + 5/x\right) \cdot dx$

$= 5\int x^3 dx + 2\int x^5 \cdot dx - 7\int x \cdot dx + \int x^{-1/2} \cdot dx + 5\int \frac{dx}{x}$

$= 5 \cdot \left(\frac{x^{3+1}}{3+1}\right) + 2 \cdot \left(\frac{x^{5+1}}{5+1}\right) - 7 \cdot \left(\frac{x^{1+1}}{1+1}\right) + \left(\frac{x^{-1/2+1}}{-1/2+1}\right) + 5 \log|x| + c$

$= 5\frac{x^4}{4} + \frac{\cancel{2}}{\cancel{6}} x^6 - \frac{7}{2}x^2 + \frac{x^{\frac{1}{2}}}{1/2} + 5 \log|x| + c$

$= \frac{5x^4}{4} + \frac{x^6}{3} - \frac{7}{2}x^2 + 2\sqrt{x} + 5 \log|x| + c$

ii) $\int (3\sin x + 4\cos x + 5\sec^2 x - 2\operatorname{cosec}^2 x)\cdot dx$

Soln:

$= 3\int \sin x\, dx + 4\int \cos x \cdot dx + 5\int \sec^2 x \cdot dx - 2\int \operatorname{cosec}^2 x \cdot dx$

$= 3(-\cos x) + 4(\sin x) + 5(\tan x) - 2(\cot x) + C$

$= -3\cos x + 4\sin x + 5\tan x - 2\cot x + C$

Q.2 : Evaluate

i) $\tan^2 x \cdot dx$

ii) $\int \frac{dx}{1+\sin x}$

Soln

i) $\int \tan^2 x \cdot dx = \int (\sec^2 x - 1)\cdot dx$

$= \int \sec^2 x \cdot dx - \int 1 \cdot dx$

$= \tan x - x + C$

ii)

Soln $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{(1-\sin x)\cdot(1+\sin x)}\cdot dx \quad \left(\because (a-b)(a+b) = a^2-b^2\right)$

$= \int \frac{1-\sin x}{1-\sin^2 x}\, dx = \int \frac{1-\sin x}{\cos^2 x}\cdot dx$

Scanned with CamScanner

$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) \cdot dx$$

$$= \int \sec^2 x \cdot dx - \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\, dx$$

$$= \int \sec^2 x \cdot dx - \int \tan x \cdot \sec x \cdot dx$$

$$= \tan x - \int \sec x \cdot \tan x \, dx$$

$$= \tan x - \sec x + c$$

Q.3 : Evaluate

i) $\displaystyle\int \frac{\sec x}{\sec x + \tan x} \cdot dx$

ii) $\displaystyle\int \sin^{-1} x (\cos x) \cdot dx$

i) Sol^n

$$\int \frac{\sec x}{\sec x + \tan x} \cdot dx$$

$$= \int \frac{\sec x (\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} \cdot dx$$

$$= \int \left(\frac{\sec^2 x - \sec x \cdot \tan x}{\sec^2 x - \tan^2 x}\right) \cdot dx \quad \left[\because \sec^2 x - \tan^2 x = 1\right]$$

$$= \int (\sec^2 x - \sec x \cdot \tan x) \cdot dx$$

$= \int \sec^2 x \cdot dx - \int \sec x \cdot \tan x \cdot dx$

$= \tan x - \sec x + c$

ii) $\int \sin^{-1}(\cos x) \cdot dx$

$= \int \cancel{\sin^{-1}} \cancel{\sin}(\pi/2 - x) \cdot dx$

$= \int (\pi/2 - x) \cdot dx =$

$= \int \pi/2 \cdot dx - \int x \cdot dx$

$= \frac{\pi}{2} x - \frac{x^{1+1}}{1+1} + c$

$= \frac{\pi x}{2} - \frac{x^2}{2} + c$

H.W

Some other Questions.

1. Evaluate

i) $\int \cot^2 x \cdot dx$ ii) $\int \left(\frac{\sin x}{1+\sin x}\right) \cdot dx$

iii) $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} \cdot dx$

iv) $\int \sin^2 x/2 \cdot dx$

Rules of integration:

i) $\frac{d}{dx}\{\int f(x)\cdot dx\} = f(x)$

ii) $\int K\cdot f(x)\cdot dx = K\int f(x)\cdot dx$

iii) $\int\{f_1(x) \pm f_2(x) \pm f_3(x) \pm \cdots \pm f_n(x)\}\cdot dx$

$= \int f_1(x)\cdot dx \pm \int f_2(x)\cdot dx \pm \int f_3(x)\cdot dx \pm \cdots \pm \int f_n(dx)$

Integration by substitution:

If given function can n't be expressed in some standard form then we transform it to some standard form by suitable substitution. This method is known as integration by substitution method.

Let $\int f(x)\cdot dx$ can be reduced one of the standard form by changing the independent variable x to t.

By substituting $x = \phi(t)$

$$\int f(x)\cdot dx = \int f(x)\cdot\left(\frac{dx}{dt}\right)\cdot dt = \int f[\phi(t)]\cdot\phi' t\cdot dt$$

where $x = \phi(t)$

Type-1

$$\int f(ax+b)\cdot dx$$

we put $(ax+b) = t \Rightarrow \frac{d}{dx}(ax+b) = \frac{d}{dx}(t)$

$\Rightarrow a\cdot dx = dt$

i.e $dx = 1/a \cdot dt$

$$= \int f(t)\cdot \frac{dt}{a} = 1/a \int f(t)\cdot dt$$

Type-2

$$\int x^{n-1}\cdot f(x^n)\cdot dx$$

put $x^n = t$

$\Rightarrow \frac{d}{dx}(x^n) = \frac{d}{dx}(t)$

$\Rightarrow n x^{n-1} = dt/dx$

$\Rightarrow n x^{n-1}\cdot dx = dt$

$\Rightarrow x^{n-1}\cdot dx = dt/n$

$$= \int f(t)\cdot dt/n = 1/n \int f(t)\cdot dt$$

Type-③ $\int \{f(x)\}^n \cdot f'(x)\cdot dx$

put $f(x) = t$

$\Rightarrow f'(x)\cdot dx = dt$

$$= \int t^n \cdot dt = \frac{t^{n+1}}{n+1} + C$$

$$= \frac{[F(x)]^{n+1}}{n+1} + C$$

Type-4

$$\int \frac{F'(x)}{F(x)} \cdot dx$$

Let $F(x) = t$

$$\Rightarrow F'(x) \cdot dx = dt$$

$$= \int \frac{dt}{t} = \log|t| + C$$

$$= \log|F(x)| + C$$

Some Important formulaes using Method of Substitution

i) $\int \tan x \cdot dx = -\log|\cos x| + C = \log|\sec x| + C$

ii) $\int \cot x \cdot dx = \log|\sin x| + C$

iii) $\int \sec x \cdot dx = \log|\sec x + \tan x| + C$

$$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

iv) $\int \operatorname{cosec} x \cdot dx = \log|\operatorname{cosec} x - \cot x| + C$

$$= \log\left|\tan \frac{x}{2}\right| + C$$

v) $\int \sin(ax+b).dx = \frac{-\cos(ax+b)}{a} + c$

vi) $\int \cos(ax+b).dx = \frac{\sin(ax+b)}{a} + c$

vii) $\int \sec^2(ax+b).dx = \frac{\tan(ax+b)}{a} + c$

viii) $\int \operatorname{cosec}^2(ax+b).dx = \frac{-\cot(ax+b)}{a} + c$

ix) $\int \sec(ax+b).\tan(ax+b).dx = \frac{\sec(ax+b)}{a} + c$

x) $\int \operatorname{cosec}(ax+b).\cot(ax+b)dx = \frac{-\operatorname{cosec}(ax+b)}{a} + c$

xi) $\int a^{mx}.dx = \frac{a^{mx}}{m\log a} + c$

xii) $\int e^{mx}.dx = \frac{e^{mx}}{m} + c$

xiii) $\int (ax+b)^n.dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

xiv) $\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + c$

11

Q:1 Integrate the followings.

i) $\int x \sin x^2 \cdot dx$

ii) $\int \frac{\operatorname{cosec}^2 x}{(1+\cot x)} \cdot dx$

iii) $\int \frac{e^{\tan^{-1} x}}{(1+x^2)} \cdot dx$

i) <u>soln</u> $\int x \sin x^2 \cdot dx$

Let $x^2 = z$. then

$2x = dz/dx$

$\Rightarrow 2x \cdot dx = dz$

$\Rightarrow x \cdot dx = dz/2$

$= \int \sin z \cdot dz/2 = \frac{1}{2} \int \sin z \cdot dz$

$= \frac{1}{2}(-\cos z) + c = -\frac{1}{2} \cos x^2 + c$

ii) <u>soln</u> $\int \left(\frac{\operatorname{cosec}^2 x}{1+\cot x}\right) \cdot dx$

put $1+\cot x = t$ $\Rightarrow d/dx(1+\cot x) = d/dx(t)$

$\Rightarrow -\operatorname{cosec}^2 x = dt/dx$

$\Rightarrow \operatorname{cosec}^2 x \cdot dx = -dt$

$$= \int -\frac{1}{t}\cdot dt = -\log|t| + C$$

$$= -\log|(1+\cot x)| + C$$

iii). Soln $\int \frac{e^{\tan^{-1}x}}{1+x^2}\cdot dx$

Put $\tan^{-1}x = t \Rightarrow d/dx(\tan^{-1}x) = d/dx(t)$

$\Rightarrow d \frac{1}{\cancel{\sec^2 x}\ 1+x^2} = dt/dx \Rightarrow \frac{1}{1+x^2} = dt/dx$

$\Rightarrow \cancel{\sec^2 x\cdot dx = dt}$

$\Rightarrow \left(\frac{1}{1+x^2}\right)\cdot dx = dt$

$$\therefore \int \frac{e^{\tan^{-1}x}}{1+x^2}\cdot dx = \int e^t\cdot dt = e^t + C$$

$$= e^{\tan^{-1}x} + C$$

Q:2. Evaluate $\int \frac{1}{(1+\tan x)}\cdot dx$

Soln:

$$\int\left(\frac{1}{1+\tan x}\right)\cdot dx = \int \frac{1}{\left(1+\frac{\sin x}{\cos x}\right)}\cdot dx$$

$$= \int\left(\frac{\cos x}{\cos x+\sin x}\right)\cdot dx = \int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{2(\cos x+\sin x)}\cdot dx$$

$$= \frac{1}{2}\int\left(\frac{\cos x+\sin x}{\cos x+\sin x}\right)\cdot dx + \frac{1}{2}\int\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\cdot dx$$

$$= \frac{1}{2}\int 1\cdot dx + \frac{1}{2}\int \frac{1}{t}\cdot dt$$

where $(\cos x+\sin x) = t$

$$\Rightarrow \frac{d}{dx}(\cos x+\sin x) = \frac{d}{dx}(t)$$

$$\Rightarrow -\sin x+\cos x = \frac{dt}{dx}$$

$$\Rightarrow (\cos x-\sin x)\cdot dx = dt$$

$$= \frac{1}{2}\int 1\cdot dx + \frac{1}{2}\int \frac{1}{t}\cdot dt$$

$$= \frac{1}{2}(x) + \frac{1}{2}\log|t| + c$$

$$= \frac{1}{2}x + \frac{1}{2}\log|\cos x+\sin x| + c$$

H.W 1. Evaluate

i) $\int(3x+5)^7\cdot dx$ ii) $\int\frac{\log x}{x}\cdot dx$

iii) $\int \cos 3x\cdot\sin x\cdot dx$

2. Evaluate $\int\frac{dx}{(\sqrt{x}+\sqrt{x+1})}$

Integration by trigonometric substitution

The following trigonometric substitution are adopted in the integrand

Form of integrand	Substitution
$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $x=a\cos\theta$
a^2+x^2	$x=a\tan\theta$ or $x=a\cot\theta$
x^2-a^2	$x=a\sec\theta$ or $x=a\,\mathrm{cosec}\,\theta$

Special integrals:

i) $\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$

ii) $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c$

iii) $\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

iv) $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+c$

v) $\int\frac{dx}{\sqrt{x^2-a^2}}=\log\left|x+\sqrt{x^2-a^2}\right|+c$

vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}|+c$

vii) $\int \sqrt{x^2+a^2}\, dx = \frac{1}{2}\left[x\sqrt{x^2+a^2}+a^2\log\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|\right]+c$

viii) $\int \sqrt{a^2-x^2}\cdot dx = \frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\sin^{-1}(x/a)\right]+c$

ix) $\int \sqrt{x^2-a^2}\cdot dx = \frac{1}{2}\left[x\sqrt{x^2-a^2}-a^2\log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|\right]+c$

x) $\int \frac{1}{x\sqrt{x^2-a^2}}\cdot dx = \frac{1}{a}\sec^{-1} x/a + c$

1. Evaluate the followings

i) $\int \frac{dx}{x^2-25}$ ii) $\int \frac{dx}{x^2+9}$

iii) $\int \sqrt{x^2+25}$ iv) $\int \sqrt{x^2-4}\cdot dx$

i) soln $\int \frac{dx}{x^2-25} = \int \frac{dx}{(x)^2-(5)^2} = \frac{1}{2(5)}\log\left|\frac{x-5}{x+5}\right|+c$

$= \frac{1}{10}\log\left|\frac{x-5}{x+5}\right|+c$

ii) $\int \frac{dx}{x^2+9} = \int \frac{dx}{(x)^2+(3)^2} = \frac{1}{3}\tan^{-1} x/3 + c$

iii) $\int(\sqrt{x^2+25})\cdot dx = \int\sqrt{(x)^2+(5)^2}\, dx$

$$= \frac{x}{2}\sqrt{x^2+25} + \frac{25}{2}\log\left|\left(\frac{x+\sqrt{x^2+25}}{5}\right)\right| + c$$

iv) $\int\sqrt{x^2-4}\cdot dx = \int\sqrt{(x)^2-(2)^2}\, dx$

$$= \frac{x}{2}\sqrt{x^2-4} - 2\log\left|\frac{x+\sqrt{x^2-4}}{2}\right| + c$$

2. Evaluate $\int\sqrt{1-x^2}\cdot dx$

<u>Soln</u>

$$\int\sqrt{1-x^2}\cdot dx$$

put $x = \sin\theta \Rightarrow d/dx(x) = d/dx(\sin\theta)$

$\Rightarrow dx = \cos\theta\cdot d\theta$

$$\therefore \int\sqrt{1-x^2}\cdot dx = \int\sqrt{1-\sin^2\theta}\cdot\cos\theta\cdot d\theta$$

$$= \int\sqrt{\cos^2\theta}\cdot\cos\theta\, d\theta = \int\cos^2\theta\cdot d\theta$$

$$= \int\left(\frac{1+\cos 2\theta}{2}\right)\cdot d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int\cos 2\theta\cdot d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{\sin 2\theta}{2} + c$$

$$= \frac{1}{2\theta} + \frac{1}{2}\sin\theta \cdot \cos\theta + C$$

$$= \frac{1}{2\sin^{-1}x} + \frac{x\sqrt{1-x^2}}{2} + C$$

Some Important Questions.

1. Evaluate $\int \sqrt{x^2+3x}\, dx$

2. Integrate $\int \frac{x+5}{\sqrt{x^2+6x-7}} \cdot dx$

3. Evaluate $\int \frac{d\theta}{\sin^2\theta \sqrt{\cot^2\theta+2}}$

4. Evaluate $\int \frac{dx}{\sqrt{2x-x^2}}$

5. Evaluate $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}} \cdot dx$

Integration by Parts:

This method is used to integrate the product of two functions.

If $f(x)$ and $g(x)$ be two integrable functions. then

$$\int \underset{1}{f(x)} \cdot \underset{2}{g(x)} \cdot dx = f(x) \cdot \int g(x) \cdot dx - \int \left\{ \frac{d}{dx} f(x) \int g(x) \cdot dx \right\} \cdot dx$$

i) We use the following preferential order for taking the first function.

Inverse $\longrightarrow$ Logarithm $\longrightarrow$ Algebraic $\longrightarrow$ Trigonometric $\longrightarrow$ Exponential. In short we write it ILATE.

ii) If one of the function is not directly integrable, then we take it as the first function.

iii) If only one function is there, i.e $\int \log x \cdot dx$ or $\int \sin^{-1} x \cdot dx$ etc. then 1 is taken as second function.

iv) If both the functions are directly integrable, then the first function is chosen in such a way that its derivative vanishes easily or the function obtained in integral sign is easily integrable.

Note:

i) Integration by parts is not applicable to product of functions in all cases.

eg. $\int \sqrt{x}\sin x \cdot dx$

ii) Normally, if any function is a power of x (polynomial in x) then we take ~~it~~ it as the Ist function.

Integral of the form:

$$\int e^x [F(x) + F'(x)] \cdot dx$$

Prove that $\int e^x [F(x) + F'(x)] \cdot dx = e^x F(x) + C$

Now $\int e^x [F(x) + F'(x)] \cdot dx = \int F(x) \cdot e^x \cdot dx + \int F'(x) \cdot e^x \cdot dx$

$$= F(x)\int e^x \cdot dx - \int \left(\frac{d}{dx}(F(x)) \cdot \int e^x \cdot dx \right) \cdot dx + \int e^x \cdot F'(x) \cdot dx$$

$$= e^x F(x) - \cancel{\int e^x \cdot F'(x) \cdot dx} + \cancel{\int e^x \cdot F'(x) \cdot dx}$$

$$= e^x F(x) + C$$

Scanned with CamScanner

i) $\int e^{ax}\sin(bx+c)dx = \frac{e^{ax}}{a^2+b^2}\{a\sin(bx+c)-b\cos(bx+c)\}+c$

ii) $\int e^{ax}\cos(bx+c)\cdot dx = \frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c)+b\sin(bx+c)\}+c$

Q.1 : Integrate

i) $\int x\cos x\,dx$ ii) $\int (\ln x)\,dx$

Sol^n i) $\int x\cos x\,dx$

$= x\int \cos x\cdot dx - \int\left(\frac{d}{dx}(x)\cdot\int\cos x\cdot dx\right)\cdot dx$

$= x\sin x - \int(1\cdot\sin x)\cdot dx$

$= x\sin x - \int \sin x\cdot dx$

$= x\sin x - (-\cos x)+c$

$= x\sin x + \cos x + c$

ii) $\int(\ln x)\cdot dx = \int(\ln x)\cdot 1\,dx$

$= \ln x\cdot\int 1\cdot dx - \int\left(\frac{d}{dx}(\ln x)\int 1\cdot dx\right)\cdot dx$

$= x\ln x - \int \frac{1}{\cancel{x}}\cdot\cancel{x}\cdot dx$

$= x\ln x - x + c$

Q:2 Evaluate: $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) \cdot dx$

Soln

$$\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) \cdot dx$$

$$= \int e^x \left(\frac{1}{1+\cos x}\right) \cdot dx + \int e^x \cdot \left(\frac{\sin x}{1+\cos x}\right) \cdot dx$$

$$= \int e^x \cdot \left(\frac{\sin x}{1+\cos x}\right) \cdot dx + \int \frac{e^x}{1+\cos x} \cdot dx$$

$$= \frac{\sin x}{1+\cos x} \int e^x \cdot dx - \int \left\{ \frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) \cdot \int e^x \cdot dx \right\} dx + \int \frac{e^x}{1+\cos x} dx$$

$$= \frac{\sin x}{1+\cos x} \cdot e^x - \cancel{\int \left(\frac{1}{1+\cos x}\right) \cdot e^x \cdot dx} + \cancel{\int \frac{e^x}{1+\cos x}} + C$$

$$= e^x \cdot \frac{\sin x}{1+\cos x} + C \qquad \left[\because \frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right) = \frac{1}{1+\cos x}\right]$$

Q:3. Evaluate $\int x \tan^{-1} x \, dx$

Soln

$$\int x \tan^{-1} x \cdot dx$$

$$= \tan^{-1} x \int x \cdot dx - \int \left\{ \frac{d}{dx}(\tan^{-1} x) \cdot \int x \cdot dx \right\} \cdot dx$$

$$= \tan^{-1} x \left(\frac{x^2}{2}\right) - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} \cdot dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int\left(\frac{(x^2+1)-1}{1+x^2}\right).dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int 1 \cdot dx + \frac{1}{2}\int \frac{1}{1+x^2} \cdot dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}x + \frac{1}{2}(\tan^{-1}x) + C$$

$$= \frac{1}{2}(x^2 \tan^{-1}x + \tan^{-1}x - x) + C$$

Q:4. Evaluate $\int \log(1+x^2).dx$

Soln: $\int \log(1+x^2) \cdot dx = \int \log(1+x^2).1 \cdot dx$

$$= \log(1+x^2) \cdot \int 1 \cdot dx - \int\left[\frac{d}{dx}(\log(1+x^2)) \cdot \int 1 \cdot dx\right] \cdot dx$$

$$= \log(1+x^2) \cdot x - \int \frac{1}{1+x^2} \cdot 2x \cdot x \cdot dx$$

$$= x\log(1+x^2) - \int \frac{2x^2}{1+x^2} \cdot dx$$

$$= x\log(1+x^2) - 2\int \frac{(1+x^2)-1}{(1+x^2)} \cdot dx$$

$$= x\log(1+x^2) - 2\int 1 \cdot dx + 2\int\left(\frac{1}{1+x^2}\right) \cdot dx$$

$$= x\log(1+x^2) - 2x + 2\tan^{-1}x + C$$

Q.5: Evaluate $\int x^2 \sin^{-1}x \, dx$

$$\int x^2 \sin^{-1}x \cdot dx = \sin^{-1}x \cdot \int x^2 \cdot dx - \int \left[\left(\frac{d \sin^{-1}x}{dx}\right) \int x^2 dx\right] dx$$

$$= \sin^{-1}x \cdot x^3/3 - \int \frac{1}{\sqrt{1-x^2}} \cdot \left(\frac{x^3}{3}\right) \cdot dx$$

$$= \frac{x^3 \sin^{-1}x}{3} - 1/3 \int \frac{x^3}{\sqrt{1-x^2}} \cdot dx$$

$$= \frac{x^3 \sin^{-1}x}{3} - 1/3 \int \frac{x \cdot x^2}{\sqrt{1-x^2}} \cdot dx$$

Putting $(1-x^2) = t^2 \Rightarrow 1-t^2 = x^2$

$$-\cancel{2}x \cdot dx = \cancel{2}t \cdot dt$$

$$x \cdot dx = t \cdot dt$$

$$= \frac{x^3 \sin^{-1}x}{3} - 1/3 \int \frac{\cancel{t}(1-t^2)}{\cancel{t}} \cdot dt$$

$$= \frac{x^3 \sin^{-1}x}{3} - 1/3 \int 1 \cdot dt + 1/3 \int t^2 \cdot dt$$

$$= \frac{x^3 \sin^{-1}x}{3} - 1/3\, t + 1/3\, t^3/3 + C$$

$$= \frac{x^3 \sin^{-1}x}{3} - 1/3 \sqrt{1-x^2} + 1/9 (1-x^2)^{3/2} + C$$

H.W Important Questions

1. Integrate $\int e^{2x} \sin x \, dx$

Scanned with CamScanner

2. Integrate $\int x \ln(1+x) \cdot dx$

3. Evaluate $\int (\log x)^2 \cdot dx$

4. Evaluate $\int x^2 \cos 2x \cdot dx$

5. Evaluate $\int \sqrt{a^2 - x^2} \cdot dx$

6. Evaluate $\int \frac{\log x}{(1+\log x)^2} \cdot dx$

Definite Integrals:

Let $F(x)$ be a function defined on the interval $[a,b]$ and $F(x)$ be its anti-derivative

Then, $$\int_a^b F(x)\cdot dx = F(b) - F(a)$$

is defined as the definite integral of $F(x)$ from $x=a$ to $x=b$.

The numbers a and b are called upper and lower limits of integration.

Properties of Definite Integrals:

i) $$\int_a^b F(x)\cdot dx = \int_a^b F(t)\cdot dt = \int_a^b F(y)\cdot dy$$

ii) $$\int_a^b F(x)\cdot dx = -\int_b^a F(x)\cdot dx$$ (order of Integration)

iii) $$\int_a^a F(x)\cdot dx = 0$$

iv) $$\int_a^b K\cdot F(x)\cdot dx = K\int_a^b F(x)\cdot dx$$

v) $$\int_a^c F(x)\cdot dx = \int_a^b F(x)\cdot dx + \int_b^c F(x)\cdot dx$$, where $a<b<c$

vi) $\int_0^a F(x).dx = \int_0^a F(a-x).dx$

vii) $\int_{-a}^{a} F(x).dx = \begin{cases} 2\int_0^a F(x).dx & \text{if } F(-x) = F(x) \\ 0, & \text{if } F(-x) = -F(x) \end{cases}$

viii) $\int_0^{2a} F(x).dx = \begin{cases} 2\int_0^a F(x).dx & \text{if } F(2a-x) = F(x) \\ 0, & \text{if } F(2a-x) = -F(x) \end{cases}$

ix) $\int_a^b F(x).dx = \int_a^b F(a+b-x).dx$

Q: 1. Evaluate the followings

i) $\int_1^2 x^3.dx$ ii) $\int_0^1 \frac{dx}{1+x^2}$

i) soln $\int_1^2 x^3.dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{1}{4}\left[2^4 - 1^4\right]$

$= \frac{1}{4}[16-1] = 15/4$

ii) $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$

$= \pi/4 - 0 = \pi/4$

Q: 2 Evaluate $\int_1^4 [x].dx$

Soln

2. $\int_1^4 [x]\,dx = \int_1^2 [x]\cdot dx + \int_2^3 [x]\cdot dx + \int_3^4 [x]\cdot dx$

$= \int_1^2 1\cdot dx + \int_2^3 2\cdot dx + \int_3^4 3\cdot dx = 1\left[x\right]_1^2 + 2\left[x\right]_2^3 + 3\left[x\right]_3^4$

$= (2-1) + 2(3-2) + 3(4-3)$

$= 1 + 2(1) + 3(1)$

$= 1+2+3 = 6$

3. Evaluate : $\int_0^1 x \log(1+x)\cdot dx$

Soln:

$\int_0^1 x\log(1+x)\cdot dx = \int_0^1 \{\log(1+x)\} x\cdot dx$

$= \left[\log(1+x)\int x\cdot dx\right]_0^1 - \int_0^1 \left[\frac{d}{dx}\log(1+x)\cdot \int x\cdot dx\right]\cdot dx$

$= \left[\frac{x^2}{2}\log(1+x)\right]_0^1 - \int_0^1 \left(\frac{1}{1+x}\right)\cdot\frac{x^2}{2}\cdot dx$

$= \frac{1}{2}\left[(1)^2\cdot\log(1+1) - 0\right] - \frac{1}{2}\int_0^1 \frac{x^2}{1+x}\cdot dx$

$= \frac{1}{2}\log 2 - \frac{1}{2}\int_0^1 \frac{(x^2-1)+1}{1+x}\cdot dx$

$= \frac{1}{2}\log 2 - \frac{1}{2}\int_0^1 \left[(x-1)+\frac{1}{1+x}\right]\cdot dx$

$$= \frac{1}{2}\log 2 - \frac{1}{2}\left[\frac{x^2}{2} - x + \log(x+1)\right]_0^1$$

$$= \frac{1}{2}\log 2 - \frac{1}{2}\left[\left(\frac{1}{2} - 1 + \log(1+1)\right) - 0\right]$$

$$= \frac{1}{2}\log 2 - \frac{1}{2}\left(\log 2 - \frac{1}{2}\right) =$$

$$= \frac{1}{2}\log 2 - \frac{1}{2}\log 2 + \frac{1}{4} = \frac{1}{4}$$

4. Evaluate $\int_0^{\pi/2} \frac{dx}{1+\cot x}$

Soln $\int_0^{\pi/2} \frac{dx}{1+\cot x} = \int_0^{\pi/2} \frac{dx}{1+\frac{\cos x}{\sin x}} = \int_0^{\pi/2} \frac{\sin x \cdot dx}{\sin x + \cos x}$

$$= \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} \cdot dx = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} \cdot dx$$

$\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I + I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} \cdot dx + \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \cdot dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\frac{\sin x + \cos x}{\sin x + \cos x}\right) \cdot dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2}$$

$$\Rightarrow 2I = (\pi/2 - 0)$$

$$\Rightarrow I = \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$$

Scanned with CamScanner

5. Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\cdot dx$

Soln: Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\cdot dx$ ...... (i)

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)}+\sqrt{\cos(\pi/2 - x)}}\cdot dx$

$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\cdot dx$ .... (ii)

Adding (i) and (ii) we have

$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\cdot dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\cdot dx$

$= \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right). dx = \int_0^{\pi/2} \cdot dx$

$\Rightarrow 2I = [x]_0^{\pi/2}$

$\Rightarrow 2I = [\pi/2 - 0]$

$\Rightarrow I = \pi/2 \cdot 1/2 = \pi/4$

ii) Evaluate $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} \cdot dx$

Soln:

Let $I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} \cdot dx$ ------ (i)

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\tan(\pi/2 - x)}}{\sqrt{\tan(\pi/2 - x)}+\sqrt{\cot(\pi/2 - x)}} \cdot dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}} \cdot dx$ ------ (ii)

Adding (i) and (ii), we have

$$I + I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} \cdot dx + \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}} \cdot dx$$

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\tan x}+\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}} \right) \cdot dx$$

$$2I = \int_0^{\pi/2} 1 \cdot dx = \left[x \right]_0^{\pi/2}$$

$$2I = [\pi/2 - 0]$$

$$\Rightarrow I = \pi/2 \cdot 1/2 = \pi/4$$

6. Evaluate $\int_0^{\pi/2} \log \tan x \cdot dx$

Soln We have

$$I = \int_0^{\pi/2} \log \tan x \cdot dx$$

$$= \int_0^{\pi/2} \log \tan(\pi/2 - x) \cdot dx$$

$$= \int_0^{\pi/2} \log \cot x \cdot dx = \int_0^{\pi/2} \log(1/\tan x) \cdot dx$$

$$= -\int_0^{\pi/2} \log \tan x \cdot dx \quad \left[\begin{array}{c} \because \log(1/\tan x) \\ \log 1 - \log \tan x \\ 0 - \log \tan x \end{array} \right]$$

$$I = -I$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Some Questions.

1. Evaluate $\int_0^{2} |x-1| \cdot dx$

2. Evaluate $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \cdot dx$

3. Evaluate $\int_0^{\pi/2} \frac{dx}{1+\tan x}$

4. Evaluate $\int_0^{\pi} \frac{x \cdot dx}{1+\sin x}$

Scanned with CamScanner

Differential Equation

Defination:

An equation that involves an independent variable, dependent variable and derivatives of the dependent variable with respect to the independent variables is called a differential equation.

e.g.

i) $dy/dx + x^2 = 2$ ii) $(dy/dx)^2 + 3y^2 = 5x$

iii) $dy/dx = 3$ iv) $d^3y/dx^3 = 0$

Order and Degree of a Differential Equation:

Order: The order of a differential equation is the order of the highest derivative occuring in the equation.

* The order of a differential equation is always a positive integer.

E.g.

i) $dy/dx + x^2 = 1$, order $= 1$

ii) $\frac{d^2y}{dx^2} + 2(dy/dx) - 1 = 0$, order $= 2$

iii) $d^3y/dx^3 = 0$, order $= 3$

iv) $d^2y/dx^2 = k[1 + (dy/dx)^2]$, order $= 2$

Degree: The degree of a differential equation is the degree (exponent) of the derivative of the highest order in the equation, after the equation is free from negative and fractional powers of the derivatives.

e.g. 1) $(dy/dx)^2 + 3y^2 = 5x$, Degree = 2

2) $dy/dx = 3/dy/dx$

$\Rightarrow d^2y/dx^2 = 3$, Degree = 2

Primitive or Solution of a Differential Equation

A Primitive or solution of a Differential equation is a functional such that this relation and the derivatives obtained from it satisfy the given differential Equation.

Ex:- $x = \cot y + c$ is the solution of the Differential Equation $dy/dx + \sin^2 y = 0$

Now $x = \cot y + c$ gives us

$dx/dy = -\mathrm{cosec}^2 y$

or $dy/dx = -\sin^2 y$

Substituting the value of dy/dx in L.H.S of differential equation, we get

$$-\sin^2 y + \sin^2 y = 0$$

$\therefore$ L.H.S = R.H.S

Thus the solution of a differential equation is a functional relation between x and y which is free from derivatives and this relation on substitution satisfies the differential equation.

General Solution:

It is that solution ~~of differential equation is a~~ which contains the number of arbitrary constant equal to the order of the differential equation. It is also called complete Primitive.

Thus in the above example the solution contains one arbitrary constant and the equation is of first order.

Particular Solution:

A particular solution of differential equation is a solution obtained from the general solution by giving of all Straight

Particular values to the arbitrary constant.

For example, putting $c = 1, 2 \cdots$ etc

we have $x = \cot y + 1$

$$x = \cot y + 2$$

which are particular solutions of the equation $dy/dx + \sin^2 y = 0$

Q.1: Show that $y = A\cos x + B\sin x$ is a solution of the differential equation $d^2y/dx^2 + y = 0$

<u>Soln:</u> we have $y = A\cos x + B\sin x$

$$\Rightarrow dy/dx = d/dx(A\cos x + B\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -A\sin x + B\cos x$$

$$\Rightarrow d^2y/dx^2 = -A\cos x - B\sin x$$

$$= -(A\cos x + B\sin x)$$

$$= -y$$

$$\Rightarrow d^2y/dx^2 + y = 0$$

which is the given differential equation.

Q 2) Find the differential equation of all straight Lines passing through the point $(1,1)$.

Soln: The equation of straight line passing through the point $(1,1)$ is $y-1=m(x-1)$, ... (i)

where m is constant.

Differentiating w.r.t x, we get $\frac{d}{dx}(y-1)=\frac{d}{dx}m(x-1)$

$$\Rightarrow dy/dx = m \cdot \frac{d}{dx}(x-1)$$

$$\Rightarrow dy/dx = m \quad \text{... (ii)}$$

Now From (i), we get

$$y-1 = dy/dx\,(x-1)$$

$$y = (x-1) \cdot dy/dx + 1$$

which is the required differential equation.

Solution of the Differential Equation of first order and first degree:

Type-(1):

Equation of the type $dy/dx = F(x)$

$$dy/dx = F(x) \Leftrightarrow dy = F(x) \cdot dx$$

$$\Leftrightarrow \int dy = \int F(x)\, dx$$

$\Leftrightarrow y = \phi(x) + c,$

where $\phi(x) = \int F(x)\cdot dx$

Ex-①

Solve:

i) $dy/dx = (x^2 + \sin 3x)$

ii) $dy/dx = \dfrac{1-\cos x}{1+\cos x}$

Soln:

i) $dy/dx = x^2 + \sin 3x \Leftrightarrow dy = (x^2 + \sin 3x)\cdot dx$

$\Leftrightarrow \int dy = \int (x^2 + \sin 3x)\cdot dx$

$\Leftrightarrow y = x^3/3 - \dfrac{\cos 3x}{3} + c$

ii) $dy/dx = \dfrac{1-\cos x}{1+\cos x} \Leftrightarrow dy/dx = \dfrac{2\sin^2 x/2}{2\cos^2 x/2} = \tan^2 x/2$

$\Leftrightarrow dy/dx = (\sec^2 x/2 - 1)$

$\Leftrightarrow dy = (\sec^2 x/2 - 1)\cdot dx$

$\Leftrightarrow \int dy = \int (\sec^2 x/2 - 1)\cdot dx$

$\Leftrightarrow y = \int \sec^2 x/2 \cdot dx - \int 1\cdot dx$

Let $x/2 = t$

$\Rightarrow 1/2 = dt/dx$

$\Rightarrow dx = 2\cdot dt$

$\Leftrightarrow y = \int \sec^2 t \cdot 2\cdot dt - x$

$\Leftrightarrow y = 2\int \sec^2 t\cdot dt - x$

$\Leftrightarrow y = 2\tan t - x + c$

$\Leftrightarrow y = 2\tan x/2 - x + c$

Type-②:

Equation of the type $dy/dx = F(y)$

$dy/dx = F(y) \Leftrightarrow dy/F(y) = dx$

$\Leftrightarrow \int \frac{dy}{F(y)} = \int dx \Leftrightarrow \log[F(y)] = x + C$

$\Leftrightarrow F(y) = e^x + C$

Ex ② i) Solve $dy/dx = y+2$ ii) $dy/dx = y^2 + 2y$

iii) $dx + \cot x \cdot dt = 0$

Soln: i) $dy/dx = y+2 \Leftrightarrow \frac{dy}{y+2} = dx$

$\Leftrightarrow \int \frac{dy}{y+2} = \int dx \Leftrightarrow \log(y+2) = x + C$

$\Leftrightarrow y+2 = e^x + C$

ii) $dy/dx = y^2 + 2y \Leftrightarrow \frac{dy}{y^2+2y} = dx$

$\Leftrightarrow \int \frac{dy}{y^2+2y} = \int dx \Leftrightarrow 1/2 \int \left(\frac{1}{y} - \frac{1}{y+2}\right) \cdot dy = \int dx$

$\Leftrightarrow \int 1/y \cdot dy - \int 1/y+2 \, dy = 2 \int dx$

$\Leftrightarrow \ln y - \ln(y+2) = 2x + C$

$\Leftrightarrow \ln\left(\frac{y}{y+2}\right) = 2x + C \Leftrightarrow \left(\frac{y}{y+2}\right) = e^{2x+C} = e^{2x} \cdot e^C$

$\Rightarrow y = (y+2)\cdot e^{2x}\cdot k$ where $k = e^c$

iii) $dx + \cot x \cdot dt = 0$

$\Rightarrow dx = -\cot x \cdot dt$

$\Rightarrow \dfrac{dx}{\cot x} = -dt$

$\Rightarrow \dfrac{dx}{\cot x} + dt = 0$

$\Rightarrow \tan x \cdot dx + dt = 0$

$\Rightarrow \int \tan x \, dx + \int dt = 0$

$\Rightarrow \ln(\sec x) + t = c$

$\Rightarrow -\ln \cos x + t = c$

$\Rightarrow \ln \cos x = t - c$

$\Rightarrow \cos x = e^{t - c}$

Type (3): Equation with variable separable:

If a given differential equation of being expressed in the form $F(x)\cdot dy + g(y)\cdot dx = 0$

i.e $F(x)\cdot dy + g(y)\, dx = 0$

$\Rightarrow F(x)\cdot dy = -g(y)\, dx$

$\Rightarrow \dfrac{dy}{g(y)} = -\dfrac{dx}{F(x)}$

Which on integration

i.e $\int \frac{dy}{g(y)} = -\int \frac{dx}{F(x)}$

$\Rightarrow \log|g(y)| = -\log|F(x)| + \log c$

$\Rightarrow \log|g(y)| + \log|F(x)| = \log c$

$\Rightarrow \log|g(y) \cdot F(x)| = \log c$

$\Rightarrow g(y) \cdot F(x) = c$

where $g(y) \cdot F(x)$ are Respectively functions of y and x is called a variable and seperable type equation. The solution of such an equation is obtained by integrating each term separately.

Q: 1 solve $dy/dx = (e^x+1)y$

Soln: $dy/dx = (e^x+1)y$

$\Rightarrow \frac{1}{y} \cdot dy = (e^x+1) \cdot dx$

$\Rightarrow \int \frac{dy}{y} = \int e^x \cdot dx + \int 1 \cdot dx$

$\Rightarrow \log y = e^x + x + c$

Scanned with CamScanner

Q:2: Solve the equation $x(1+y)^2 dx + y(1+x^2) dy = 0$

Given equn $\quad x(1+y)^2 . dx + y(1+x^2) . dy = 0$

$$\Rightarrow x(1+y)^2 dx = -y(1+x^2) dy$$

$$\Rightarrow \left(\frac{x}{1+x^2}\right) . dx = \left(\frac{-y}{1+y^2}\right) . dy$$

$$\Rightarrow \left(\frac{x}{1+x^2}\right) . dx + \left(\frac{y}{1+y^2}\right) dy = 0$$

Integrating, we get

$$\int \frac{x}{1+x^2} . dx + \int \frac{y}{1+y^2} . dy = 0$$

or $\frac{1}{2}\int \frac{2x}{1+x^2} . dx + \frac{1}{2}\int \frac{2y}{1+y^2} . dy = 0$

or $\frac{1}{2}\int \frac{dt}{t} + \frac{1}{2}\int \frac{du}{u} = 0$

Let $1+x^2 = t$
$2x \, dx = dt$
Let $1+y^2 = u$
$2y \, dy = du$

or $\frac{1}{2}\log t + \frac{1}{2}\log u = c$

or $\log(1+x^2) + \log(1+y^2) = 2c$

or $\log(1+x^2)(1+y^2) = 2c = \log K$ (say)

where $\log K = 2c$

$\therefore (1+x^2)(1+y^2) = K$

which is the required solution

Q:3 solve the differential equation $\frac{dx}{dy} + \sqrt{\frac{1-x^2}{1-y^2}} = 0$

Soln: Given equation $\frac{dx}{dy} + \sqrt{\frac{1-x^2}{1-y^2}} = 0$

$$\Rightarrow \frac{dx}{dy} = -\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = -\frac{dy}{\sqrt{1-y^2}}$$

$$\Rightarrow \frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0$$

Integrating, we get

$$\int \frac{dx}{\sqrt{1-x^2}} + \int \frac{dy}{\sqrt{1-y^2}} = C$$

or $\sin^{-1}x + \sin^{-1}y = c$

which is the required solution.

Q:4 solve $(1+x^2)dy + (1+y^2)dx = 0$

Soln: $(1+x^2)\cdot dy + (1+y^2)\cdot dx = 0$

$(1+x^2)dy = -(1+y^2)dx$

$$\Rightarrow -\frac{dx}{1+x^2} = \frac{dy}{1+y^2}$$

which on integrating

$$-\int \frac{dx}{1+x^2} = \int \frac{dy}{1+y^2}$$

$-\tan^{-1}x = \tan^{-1}y + \tan^{-1}c$, where $\tan^{-1}c$ is a constant

$\Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}c$

$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}c$

$\Rightarrow \frac{x+y}{1-xy} = c$

Solution of Linear differential Equation:

A Linear differential equation of the first order of the form $dy/dx + Py = Q$.

where P and Q are the functions of x

consider the differential eqn

$$dy/dx + Py = Q$$

Now integrating factor (IF) $= e^{\int P\cdot dx}$

$\therefore$ solution is $y \cdot e^{\int P\cdot dx} = \int Q \cdot e^{\int P\cdot dx} dx + c$

i.e $y(IF) = \int Q(IF)\cdot dx + c$

Some questions.

1. Solve the differential equation:

$$x^2(y-1)\,dx + y^2(x-1)\,dy = 0$$

2. Solve the differential equation

$$(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$$

3. Solve $\;dy/dx + \left(\frac{1+\cos 2y}{1-\cos 2x}\right) = 0$

4. Find the differential equation of the family of the curves $y = e^x(A\cos x + B\sin x)$